

# Radical Rings with Soluble Adjoint Groups

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An associative ring  $R$ , not necessarily with an identity, is called radical if it coincides with its Jacobson radical, which means that the set of all elements of  $R$  forms a group denoted by  $R^\circ$  under the circle operation  $r \circ s = r + s + rs$  on  $R$ . It is proved that every radical ring  $R$  whose adjoint group  $R^\circ$  is soluble must be Lie-soluble. Moreover, if the commutator factor group of  $R^\circ$  has finite torsion-free rank, then  $R$  is locally nilpotent. © 2002 Elsevier Science

*Key Words:* radical ring; adjoint group; Lie-soluble ring; soluble group.

## 1. INTRODUCTION

Let  $R$  be an associative ring, not necessarily with an identity element. The set of all elements of  $R$  forms a semigroup with neutral element  $0 \in R$  under the operation  $r \circ s = r + s + rs$  for all  $r$  and  $s$  of  $R$ . The group of all invertible elements of this semigroup is called the *adjoint group* of  $R$  and is

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denoted by  $R^\circ$ . Clearly, if  $R$  has an identity 1, then  $1 + R^\circ$  coincides with the multiplicative group  $R^*$  of  $R$  and the map  $r \mapsto 1 + r$  with  $r \in R$  is an isomorphism from  $R^\circ$  onto  $R^*$ .

Following Jacobson [5], a ring  $R$  is called *radical* if  $R = R^\circ$ , which means that  $R$  coincides with its Jacobson radical. Obviously such a ring does not have an identity element. Recall that every associative ring  $R$  can be considered as a Lie ring under the Lie multiplication  $[r, s] = rs - sr$  for all  $r, s \in R$ . For any additive subgroups  $V$  and  $W$  of  $R$ , we denote by  $[V, W]$  the additive subgroup of  $R$  generated by all Lie commutators  $[v, w]$  with  $v \in V$  and  $w \in W$ . Then  $V$  is a *Lie ideal* of  $R$  if  $[V, R] \subseteq V$ .

The derived chain of a Lie ring  $R$  is defined inductively as  $\delta_0(R) = R$  and  $\delta_{n+1}(R) = [\delta_n(R), \delta_n(R)]$  for each integer  $n \geq 0$ . The ring  $R$  is called *Lie-soluble of length at most  $m$*  if  $\delta_m(R) = 0$ . Lie-soluble rings of length at most 2 are called *Lie-metabelian*. If  $r_1, r_2, \dots$  are elements of  $R$ , the Lie commutators  $[r_1, \dots, r_{n+1}]$  are defined inductively by  $[r_1, \dots, r_{n+1}] = [[r_1, \dots, r_n], r_{n+1}]$  for all  $n \geq 2$ . The ring  $R$  is called *Lie-nilpotent of class at most  $n$*  if  $[r_1, \dots, r_{n+1}] = 0$  for all  $r_1, \dots, r_{n+1}$  of  $R$ . Recall also that soluble groups of derived length  $m$  and nilpotent group of class  $n$  are defined in a corresponding way where the usual group commutator replaces the Lie commutator. We use brackets to denote Lie commutators and parentheses to denote group commutators.

It was shown by Jennings [6] that a radical ring  $R$  is Lie-nilpotent if and only if its adjoint group is nilpotent and by Du [4] that the nilpotency classes of both structures coincide. Furthermore, we have proved in [3] that  $R$  is an  $n$ -Engel ring for some positive integer  $n$  if and only if  $R^\circ$  is an  $m$ -Engel group for some positive integer  $m$  depending only on  $n$ . Here the ring  $R$  is  $n$ -Engel if  $[r, s, \dots, s] = 0$  for each pair of elements  $r$  and  $s$  in  $R$  where  $s$  appears exactly  $n$  times;  $m$ -Engel groups are defined correspondingly. It was proved by Dickenschied and the authors in [1] that  $m$ -Engel subgroups of the adjoint group of a radical ring are always locally nilpotent.

There are several results on Lie-soluble rings (see [2] for details). For instance, it was proved by Zalesskii and Smirnov [12] and independently by Sharma and Srivastava [9] that every Lie-soluble ring  $R$  has a nilpotent ideal  $I$  whose factor ring  $R/I$  is centre-by-metabelian as a Lie ring. However, the adjoint group  $R^\circ$  of  $R$  need not be soluble in this case. An example is the ring  $M_2(R)$  of  $(2 \times 2)$ -matrices over any infinite commutative domain  $R$  of characteristic 2. On the other hand, Krasil'nikov [8] and independently Sharma and Srivastava [10] proved that the adjoint group of every Lie-metabelian ring is metabelian.

The question arises as to *whether every radical ring with soluble adjoint group must be Lie-soluble*. One of the first results in this connection is contained in the dissertation of Smirnov [11], where a positive answer was obtained for a nil algebra over an infinite field. Furthermore, in [8] it is

proved that every nil ring with metabelian adjoint group is Lie-metabelian. The following theorem yields a complete answer to the above question.

We will say that an ideal  $I$  of  $R$  is a *commutative ideal* if  $[I, I] = 0$ .

**THEOREM A.** *Let  $R$  be a radical ring. Then the adjoint group  $R^\circ$  is soluble of derived length at most  $n$  for some positive integer  $n$  if and only if the following statements hold:*

- (1)  *$R$  is Lie-soluble of length at most  $l$  for some positive integer  $l$  depending only on  $n$ , and*
- (2) *there exists a chain*

$$0 = I_0 \subseteq I_1 \subseteq \cdots \subseteq I_m = R$$

*of ideals of  $R$  such that every factor  $I_i/I_{i-1}$  is generated by commutative ideals of  $R/I_{i-1}$  for  $1 \leq i \leq m$  and the length  $m$  of such a shortest chain of  $R$  depends only on  $n$ .*

Our proof of this theorem is realized in two steps. First we prove that prime radical rings with soluble adjoint groups are commutative and then we reduce the general case to the special case where such a ring is generated by its commutative ideals.

Recall that the *Levitzki radical* of a ring is its unique maximal locally nilpotent ideal.

**THEOREM B.** *Let  $R$  be a radical ring whose adjoint group  $R^\circ$  is soluble of derived length  $n$  for some positive integer  $n$  and  $L$  be the Levitzki radical of  $R$ . Then there exist positive integers  $k, l$ , and  $m$  depending only on  $n$  such that the following statements hold.*

- (1) *The ring  $R$  satisfies the identity  $[x, y]^k = 0$  for all  $x, y \in R$ .*
- (2) *The factor ring  $R/L$  is commutative and  $L$  is an  $l$ -Engel ring.*
- (3) *The derived subgroup of  $R^\circ$  is an  $m$ -Engel group.*

An abelian group  $G$  is said to be of *finite torsion-free rank* if it has a finitely generated torsion-free subgroup  $A$  such that the factor group  $G/A$  is periodic. Since every commutative radical ring whose adjoint group has finite torsion-free rank is locally nilpotent (see for instance [2, Theorem 3.4]), the following assertion giving an affirmative answer to a question posed in [2, Sect. 3.4] is an immediate consequence of Statement (1) of Theorem B.

**COROLLARY.** *If  $R$  is a radical ring whose adjoint group  $R^\circ$  is soluble and the commutator factor group of  $R^\circ$  has finite torsion-free rank, then  $R$  is locally nilpotent. In particular, every radical ring whose adjoint group is finitely generated and soluble must be nilpotent.*

The notation is mainly standard. Throughout the paper  $\mathfrak{R}$  denotes the ring obtained by adjoining a formal unity 1 to a ring  $R$  when  $R$  has no unity, and  $\mathfrak{R} = R$  otherwise. Recall that  $R$  is an ideal in  $\mathfrak{R}$  and every element of  $\mathfrak{R}$  can uniquely be written in the form  $m + r$  with  $m \in \mathbb{Z}$  and  $r \in R$ . If  $G$  is a subgroup of the multiplicative group  $\mathfrak{R}^*$  of  $\mathfrak{R}$ , the  $i$ th term of the derived series of  $G$  will be denoted by  $\delta_i(G)$ . Recall that  $\delta_0(G) = G$  and  $\delta_{i+1}(G) = (\delta_i(G), \delta_i(G))$  for  $i \geq 0$ . For a subset  $S$  of a radical ring  $R$ , we define the *radical join* of  $S$  in  $R$  to be the smallest radical subring of  $R$  containing  $S$ .

## 2. PRIME RADICAL RINGS

Recall that a non-zero element  $r$  of a ring  $R$  is a *zero divisor* of  $R$  if there is a non-zero element  $s$  in  $R$  such that  $rs = 0$  or  $sr = 0$ .

**LEMMA 2.1.** *Let  $R$  be a radical ring and  $A$  an abelian normal subgroup of the adjoint group  $R^\circ$  of  $R$ . If  $A$  has no zero divisors of  $R$ , then  $A$  lies in the centre of  $R$ .*

*Proof.* Let  $\widehat{A}$  be the radical join of  $A$ . It is clear that an element  $r$  of  $R$  centralizes  $\widehat{A}$  if and only if  $[r, A] = 0$  and that  $\widehat{A}^\circ$  is an abelian normal subgroup of  $R^\circ$ . Assume that  $[r, A] \neq 0$  for some  $r \in R$  and show that  $(1+r)\widehat{A} \cap A \neq 0$ .

Indeed, since  $1+A$  is a normal subgroup of  $\mathfrak{R}^*$ , for every two elements  $a, b \in A$  we have  $c = b(1+r, 1+a) \in \widehat{A}$  and  $(1+(1+r)b, 1+a) = (1+(1+r)b)^{-1}(1+(1+r)c) \in 1+A$ . Therefore  $(1+(1+r)c) = (1+(1+r)b)(1+d)$  for some  $d \in A$  from which it follows that  $(1+r)(c-b-bd) = d \in (1+r)\widehat{A} \cap A$ . Hence, if  $(1+r)\widehat{A} \cap A = 0$ , then  $d = 0$  and so  $(1+(1+r)b, 1+a) = 1$ . Thus  $0 = [(1+r)b, a] = [rb, a] = [r, a]b$  which means that  $[r, a]A = 0$ . Therefore  $[r, a] = 0$  and so  $[r, A] = 0$ , contrary to the assumption.

Hence  $(1+r)\widehat{A} \cap A \neq 0$  and so  $\widehat{A} \cap A(1+r)^{-1} \neq 0$ . Thus there exists a non-zero element  $b \in A$  such that  $b(1+r)^{-1} \in \widehat{A}$ . This implies that  $0 = [b(1+r)^{-1}, A] = b[(1+r)^{-1}, A]$  and so  $[(1+r)^{-1}, A] = 0$ . Therefore  $[r, A] = 0$ , contrary to the assumption. The lemma is proved.

**COROLLARY 2.2.** *Let  $R$  be a radical ring whose adjoint group  $R^\circ$  is soluble. If  $R$  has no zero divisors, then  $R$  is commutative.*

*Proof.* It is not difficult to see that every soluble group whose abelian normal subgroups are central must be abelian, so that  $R^\circ$  and thus  $R$  are commutative by Lemma 2.1.

Recall that, for a ring  $R$ , a right (left) ideal  $P$  of  $R$  is called a *right (left) annihilator* of  $R$  if there exists a subset  $S$  of  $R$  such that  $P$  is the right (left) annihilator of  $S$  in  $R$ . The ring  $R$  is called a *right Goldie ring* if  $R$  satisfies the maximum condition for right annihilators of  $R$  and does not contain any direct sum of infinitely many right ideals of  $R$ . A left Goldie ring is defined similarly. A *Goldie ring* is a ring which is both a right and a left Goldie ring.

A ring  $R$  is said to be *prime* if for every two non-zero ideals  $P$  and  $Q$  of  $R$  the product  $PQ$  is non-zero. It is easy to see that this condition holds if and only if, for any elements  $a, b \in R$ , the equality  $aRb = 0$  implies  $a = 0$  or  $b = 0$ .

The following lemma is a group commutator version of [7, Lemma 7.3.2] for radical rings and will be proved by a similar approach.

LEMMA 2.3. *Let  $R$  be a prime radical ring with soluble adjoint group  $R^\circ$ . Then  $R$  is a Goldie ring.*

*Proof.* Let  $S$  be a right ideal of  $R$  and  $K$  its left annihilator; i.e.,  $K = \{x \mid xS = 0, x \in S\}$ . Clearly  $K$  is an ideal of  $S$  and  $K^2 = 0$ . Show first that if  $S$  is a direct sum of  $2^m$  non-zero right ideals  $P_1, \dots, P_{2^m}$  of  $R$ , then the  $m$ th commutator subgroup of the adjoint group  $S^\circ$  of  $S$  is not contained in  $K$ .

Suppose the contrary, and let  $S$  be a counterexample with minimal  $m$ . If  $m = 0$ , then  $S = K$  and therefore  $RSRS = 0$ , contrary to the fact that  $S \neq 0$  and  $R$  is prime. Hence  $m \geq 1$  and thus  $P = P_1 + \dots + P_{2^{m-1}}$  and  $Q = P_{2^{m-1}+1} + \dots + P_{2^m}$  are non-zero right ideals of  $R$ . Let  $A$  and  $B$  denote the  $(m-1)$ th commutator subgroups of the adjoint groups  $P^\circ$  and  $Q^\circ$ , respectively. By the choice of  $S$ , we have  $AP \neq 0 \neq BQ$  and the  $(m-1)$ th commutator subgroup of  $S^\circ$  is abelian modulo  $K$ . Therefore  $[A, B] \subseteq K$  and so  $AB + K = BA + K$ . Obviously  $K = (K \cap P) + (K \cap Q)$  because  $S = P + Q$  and  $P \cap Q = 0$ . Since  $AB \subseteq P$  and  $BA \subseteq Q$ , it follows that  $AB \subseteq K$  and so  $(1 + P)AB \subseteq K$ . Hence  $APB \subseteq K$  because  $(1 + P)A = A(1 + P)$ , so that  $APBS = 0$ . As  $R$  is prime and  $BS \supseteq BQ \neq 0$ , this implies that  $AP = 0$ , contrary to the choice of  $S$ .

Thus the derived length of  $S^\circ$  must be at least  $m + 1$  and this means that every direct sum of non-zero right ideals of  $R$  has less than  $2^n$  summands, where  $n$  is the derived length of  $R^\circ$ . Show now that any ascending chain of right annihilators of  $R$  is stabilized in at most  $2^n + 1$  steps. Obviously the same conclusions hold for direct sums of non-zero left ideals and chains of left annihilators.

Indeed, otherwise there is a strongly ascending chain

$$0 \subset K_1 \subset K_2 \subset \dots \subset K_{2^n} \subset R$$

of right annihilators of  $R$ . Therefore there exists a strongly descending chain

$$R \supset L_1 \supset L_2 \supset \dots \supset L_{2^n} \supset 0$$

of left ideals of  $R$  such that  $K_i = \{r \mid L_i r = 0, r \in R\}$ . Clearly, for all  $1 \leq i \leq j \leq 2^n$ , we have  $L_i K_{i+1} \neq 0$  and  $L_j K_i = 0$ , so that  $(K_i L_i \cdots K_j L_j)^2 = 0$ . Hence, if  $x \in K_i L_i, \dots, K_j L_j$  and  $y \in K_k L_k, \dots, K_l L_l$  with  $1 \leq i \leq j \leq k \leq l \leq 2^n$ , then  $(1+x, 1+y) = (1-x)(1-y)(1+x)(1+y) = 1+xy$ . Next, for each  $i$  we take any element  $x_i \in L_i K_i$  and consider the elements  $\mu_n(x_1, \dots, x_{2^n})$ , where  $\mu_0(1+x) = 1+x$  and  $\mu_{k+1}(x_1, \dots, x_{2^{k+1}}) = (\mu_k(x_1, \dots, x_{2^k}), \mu_k(x_{2^k+1}, \dots, x_{2^{k+1}}))$  for each integer  $k \geq 0$ . Arguing by induction, it is easy to see that  $\mu_n(x_1, \dots, x_{2^n}) = 1 + x_1 x_2 \cdots x_{2^n}$ . Since  $R^\circ$  is soluble of derived length  $n$  and so  $\mu_n(x_1, \dots, x_{2^n}) = 1$  for all  $x_i \in L_i K_i$ , this implies  $x_1 x_2 \cdots x_{2^n} = 0$  and therefore

$$K_1 L_1 K_2 L_2 \cdots K_{2^n} L_{2^n} = 0.$$

As  $R$  is prime, it follows that  $K_1 = 0$  or  $L_{2^n} = 0$ , contrary to the assumption. Thus, the length of every strongly ascending chain of right annihilators of  $R$  does not exceed  $2^n + 1$ , and the lemma is proved.

Recall that a subring  $R$  of a ring  $A$  with an identity element 1 is called a *right order* in  $A$  if every element of  $A$  has the form  $rs^{-1}$  for some elements  $r, s \in R$  such that  $s$  is invertible in  $S$ .

**LEMMA 2.4.** *Let  $R$  be a prime radical ring. If the adjoint group  $R^\circ$  is soluble, then  $R$  is commutative.*

*Proof.* Since  $R$  is a Goldie ring by Lemma 2.3, it is a left order in a simple artinian ring  $S$  by Goldie's theorem on prime rings (see [5, Appendix B, p. 268]). The ring  $S$  can be viewed as a complete matrix ring  $\mathbf{M}_n(D)$  over a division ring  $D$  by the Wedderburn–Artin theorem [5, Chap. III, p. 40]. Furthermore, by a theorem of Faith and Utumi (see [5, Appendix B, p. 272]), there exists a left order  $C$  in  $D$  such that  $\mathbf{M}_n(C)$  is a subring of  $R$ . Clearly the adjoint group of  $\mathbf{M}_n(C)$  is a subgroup of  $R^\circ$  and so is soluble. Moreover, every finite subgroup of this group is nilpotent by [1, Theorem A]. But this implies that  $n = 1$  because otherwise every non-zero commutative subring  $A$  of  $C$  is infinite and so the adjoint group of  $\mathbf{M}_n(A)$  cannot be soluble. Therefore  $R$  is a subring of  $D$  and hence  $R$  is commutative by Corollary 2.2. The lemma is proved.

### 3. STRONGLY LIE-HYPERABELIAN RINGS

A ring  $R$  is *strongly Lie-hyperabelian* if there exists an ascending series

$$0 = R_0 \subseteq R_1 \subseteq \cdots \subseteq R_\alpha \subseteq R_{\alpha+1} \subseteq \cdots \subseteq R_\gamma = R$$

of ideals of  $R$  whose factors  $R_{\alpha+1}/R_\alpha$  are commutative for all ordinals  $\alpha$ .

LEMMA 3.1. *Let  $R$  be a strongly Lie-hyperabelian ring and let  $x_1, y_1, y_2, \dots$  and  $z_1, z_2, \dots$  be two sequences of elements of  $R$ . If*

$$x_{i+1} = [y_{2i-1}x_i y_{2i}, z_{2i-1}x_i z_{2i}]$$

*for each integer  $i \geq 1$ , then there exists an integer  $m \geq 1$  such that  $x_m = 0$ .*

*Proof.* Suppose the contrary and let  $x_1, y_1, y_2, \dots$  and  $z_1, z_2, \dots$  be two sequences of elements of  $R$  such that  $x_{i+1} = [y_{2i-1}x_i y_{2i}, z_{2i-1}x_i z_{2i}] \neq 0$  for all  $i \geq 1$ . Without loss of generality we may assume that  $R$  is generated by all elements of these sequences, so that  $R$  is countable. Let  $0 = R_0 \subseteq R_1 \subseteq \dots \subseteq R_\alpha \subseteq R_{\alpha+1} \subseteq \dots \subseteq R_\gamma = R$  be an ascending series of ideals of  $R$  with commutative factors. It is clear that for each  $x_i$  there exists some ordinal  $\alpha_i$  such that  $x_i \in R_{\alpha_i+1} \setminus R_{\alpha_i}$ . Since the factor  $R_{\alpha_i+1}/R_{\alpha_i}$  is commutative, this implies

$$x_{i+1} = [y_{2i-1}x_i y_{2i}, z_{2i-1}x_i z_{2i}] \in [R_{\alpha_i+1}, R_{\alpha_i+1}] \subseteq R_{\alpha_i},$$

so that  $\alpha_{i+1} < \alpha_i$ . Hence  $\alpha_1 > \alpha_2 > \dots$  is an infinite descending chain of ordinals. This contradiction shows that  $x_m = 0$  for some  $m \geq 1$ , and the lemma is proved.

We will say that an ideal  $I$  of  $R$  is a *null ideal* if  $I^2 = 0$ .

LEMMA 3.2. *Let  $A$  be a commutative ideal of a ring  $R$ . Then*

- (1)  *$A[A, R] = 0$  and  $A^2$  is contained in the centre of  $R$ ,*
- (2) *if  $B$  and  $C$  are additive subgroups of  $R$  such that  $B \subseteq [A, R]$  and  $C$  generates a null ideal of  $R$ , then  $(1 + B, 1 + C) = 1 + [B, C]$ , and*
- (3) *if  $R$  is a radical ring, then  $(1 + A, 1 + R) = 1 + [A, R]$ .*

*Proof.* If  $a, c \in A$  and  $r \in R$ , then  $(ar)c = c(ar) = a(cr)$  and therefore  $a[c, r] = 0$ . Furthermore,  $acr = cra = rac$ , and so (1) holds. From this it follows that  $[A, R]^2 = 0$  and so  $[B, C]^2 = (B + C)[B, C] = 0$ . Thus, for any  $b \in B$  and  $d \in C$ , we have  $(1 + b, 1 + d) = 1 + [b, d]$ , so that (2) holds.

Finally, since  $(1 + a, 1 + r) = 1 + (1 + a)^{-1}(1 + r)^{-1}[a, r]$  for any  $a \in A$  and  $r \in R$ , it follows from (1) that  $(1 + a, 1 + r) = 1 + (1 + r)^{-1}[a, r] = 1 + [(1 + r)^{-1}a, r] \in 1 + [A, R]$ . Hence  $(1 + A, 1 + R) \subseteq 1 + [A, R]$  because  $(1 + [a, r])(1 + [c, s]) = 1 + [a, r] + [c, s]$  for any  $c \in A$  and  $s \in R$ . Conversely, if  $b = (1 + r)a$ , then  $b \in A$  and  $[a, r] = [(1 + r)^{-1}b, r] = (1 + r)^{-1}[b, r] = (1 + b)^{-1}(1 + r)^{-1}[b, r]$ , so that  $1 + [a, r] = (1 + b, 1 + r)$ . Thus  $1 + [A, R] \subseteq (1 + A, 1 + R)$  and this gives (3). The lemma is proved.

Let  $B, B_1, B_2, \dots$  be an infinite sequence of ideals of a radical ring  $R$ . The following sequences of commutator subgroups of  $1 + R$  and Lie-commutator ideals of  $R$ , regarded as a Lie ring, are defined inductively

as  $\mu_0(1+B) = 1+B$ ,

$$\begin{aligned} & \mu_{k+1}(1+B_1, \dots, 1+B_{2^{k+1}}) \\ &= (\mu_k(1+B_1, \dots, 1+B_{2^k}), \mu_k(1+B_{2^k+1}, \dots, 1+B_{2^{k+1}})), \end{aligned}$$

and similarly  $\nu_0(B) = B$ ,

$$\nu_{k+1}(B_1, \dots, B_{2^{k+1}}) = [\nu_k(B_1, \dots, B_{2^k}), \nu_k(B_{2^k+1}, \dots, B_{2^{k+1}})],$$

for each integer  $k \geq 0$ .

LEMMA 3.3. *Let  $R$  be a radical ring and  $B_1, B_2, \dots$  an infinite sequence of commutative ideals of  $R$ . Then*

$$\mu_k(1+B_1, \dots, 1+B_{2^k}) = 1 + \nu_k(B_1, \dots, B_{2^k})$$

for any integer  $k \geq 0$ .

*Proof.* Obviously the equality holds for  $k = 0$ . By the induction hypothesis

$$\mu_{k-1}(1+B_1, \dots, 1+B_{2^{k-1}}) = 1 + \nu_{k-1}(B_1, \dots, B_{2^{k-1}}),$$

so that

$$\begin{aligned} & \mu_k(1+B_1, \dots, 1+B_{2^k}) \\ &= (1 + \nu_{k-1}(B_1, \dots, B_{2^{k-1}}), 1 + \nu_{k-1}(B_{2^{k-1}+1}, \dots, B_{2^k})) \\ &= 1 + [\nu_{k-1}(B_1, \dots, B_{2^{k-1}}), \nu_{k-1}(B_{2^{k-1}+1}, \dots, B_{2^k})] \\ &= 1 + \nu_k(B_1, \dots, B_{2^k}) \end{aligned}$$

by Lemma 3.2. The lemma is proved.

LEMMA 3.4. *Let  $R$  be a radical ring whose adjoint group is soluble of derived length  $n$ . If  $R$  is generated by its commutative ideals, then  $R$  is Lie-soluble of length  $n$ .*

*Proof.* Clearly, without loss of generality we may assume that

$$R = A_1 + \dots + A_m$$

is a sum of finitely many commutative ideals  $A_1, \dots, A_m$  for some integer  $m \geq 2^n$ . Then

$$\delta_1(R) = [R, R] = \sum_{1 \leq i_1 \leq m} \sum_{1 \leq i_2 \leq m} \nu_1(A_{i_1}, A_{i_2}),$$

and by induction it is easily verified that

$$\delta_k(R) = \sum_{1 \leq i_1 \leq m} \dots \sum_{1 \leq i_{2^k} \leq m} \nu_k(A_{i_1}, \dots, A_{i_{2^k}})$$

for each integer  $k \geq 0$ .



On the other hand,  $1 + R$  as a subgroup of  $\mathfrak{R}^*$  is generated by its abelian normal subgroups  $1 + A_1, \dots, 1 + A_m$ , so that

$$1 + R = (1 + A_1) \cdots (1 + A_m).$$

Therefore the derived subgroup  $\delta_1(1 + R) = (1 + R, 1 + R)$  is generated by all commutator subgroups  $(1 + A_i, 1 + A_j)$  with  $1 \leq i \leq j \leq n$ , each of which is an abelian normal subgroup of  $1 + R$ . Hence

$$\delta_1(1 + R) = \prod_{1 \leq i_1 \leq m} \prod_{1 \leq i_2 \leq m} \mu_1(1 + A_{i_1}, 1 + A_{i_2}).$$

Arguing inductively, for each integer  $k \geq 0$  we obtain that

$$\delta_k(1 + R) = \prod_{1 \leq i_1 \leq m} \cdots \prod_{1 \leq i_{2k} \leq m} \mu_k(1 + A_{i_1}, \dots, 1 + A_{i_{2k}}).$$

Since  $\mu_k(1 + A_{i_1}, \dots, 1 + A_{i_{2k}}) = 1 + \nu_k(A_{i_1}, \dots, A_{i_{2k}})$  by Lemma 3.3, this implies that

$$\delta_k(1 + R) = \prod_{1 \leq i_1 \leq m} \cdots \prod_{1 \leq i_{2k} \leq m} (1 + \nu_k(A_{i_1}, \dots, A_{i_{2k}}))$$

for every  $k \geq 0$ . Thus,  $\delta_k(1 + R) = 1$  for some integer  $k \geq 0$  if and only if  $\delta_k(R) = 0$  for the same  $k$ , so that  $R$  is a Lie-soluble ring of length  $n$ . The lemma is proved.

**LEMMA 3.5.** *Let  $R$  be a radical ring whose adjoint group is soluble of derived length  $n$ . For every two infinite sequences  $x_1, y_1, y_2, \dots$  and  $z_1, z_2, \dots$  of elements of  $R$  and each integer  $i \geq 1$ , we put  $x_{i+1} = [y_{2i-1}x_i y_{2i}, z_{2i-1}x_i z_{2i}]$ . If there exists an integer  $m \geq 1$  such that  $x_{m+1} = 0$  for any choice of such sequences in  $R$ , then the ring  $R$  is Lie-soluble of length at most  $mn$  and there exists a finite chain*

$$0 = I_0 \subseteq \cdots \subseteq I_i \subseteq I_{i+1} \subseteq \cdots \subseteq I_m = R$$

*of ideals of  $R$  such that every factor  $I_{i+1}/I_i$  is generated by all commutative ideals of the factor ring  $R/I_i$  for  $0 \leq i \leq m-1$ .*

*Proof.* Put  $I_0 = 0$  and let  $I_1$  be the sum of all commutative ideals of  $R$ . Then  $I_1$  is an ideal of  $R$  which, regarded as a ring, is Lie-soluble of length at most  $n$  by Lemma 3.4. Since  $0 = x_{m+1} = [y_{2m-1}x_my_{2m}, z_{2m-1}x_mz_{2m}]$  for any elements  $y_{2m-1}, y_{2m}, z_{2m-1}, z_{2m}$  of  $R$ , the ideal  $\mathfrak{R}x_m\mathfrak{R}$  of  $R$  generated by  $x_m$  is commutative and so  $x_m \in I_1$ . If  $m = 1$ , then  $R = I_1$  and we are done. By induction on  $m$ , the factor ring  $R/I_1$  is Lie-soluble of length  $(m-1)n$  and there exists a finite chain of ideals  $I_i/I_1$  of  $R/I_1$  with  $1 \leq i \leq m-1$  which satisfies the conclusion of the lemma. Therefore  $R$  is a Lie-soluble ring of length  $mn$  and its ideals  $I_i$  with  $0 \leq i \leq m-1$  form the required finite chain. The lemma is proved.

## 4. PROOF OF THEOREMS A AND B

Recall first that the *lower nil radical* of Baer is the last term of the ascending series  $0 = R_0 \subseteq \cdots \subseteq R_{\alpha-1} \subseteq R_\alpha \subseteq \cdots$  of ideals of  $R$  each of which is defined inductively as follows: the factor  $R_\alpha/R_{\alpha-1}$  is generated by all nilpotent ideals of the factor ring  $R/R_{\alpha-1}$  for every non-limit ordinal  $\alpha \geq 1$  and  $R_\alpha = \bigcup_{\beta < \alpha} R_\beta$  for every limit ordinal  $\alpha$  (see [5, Chap. VIII, p. 194]).

*Proof of Theorem A.* Let the adjoint group  $R^\circ$  of  $R$  be soluble of derived length  $n$  for some positive integer  $n$ . Then every prime factor ring of  $R$  is commutative by Lemma 2.4. Since the lower nil radical  $L$  of  $R$  is the intersection of all ideals of  $R$  modulo which  $R$  is prime (see [5, Chap. VIII, Theorem 1, p. 196]), the factor ring  $R/L$  is also commutative. Therefore  $R$  has an ascending series of ideals of  $R$  whose factors are commutative and so is strongly Lie-hyperabelian. Hence, by Lemma 3.1, if  $x_1, y_1, y_2, \dots$  and  $z_1, z_2, \dots$  are two sequences of elements of  $R$  and

$$(*) \quad x_{i+1} = [y_{2i-1}x_iy_{2i}, z_{2i-1}x_iz_{2i}]$$

for each integer  $i \geq 1$ , then there exists an integer  $m \geq 1$  such that  $x_m = 0$ . Show that the least integer  $m$  with this property does not depend on the choice of such sequences in  $R$  and is bounded by a function of  $n$ .

Indeed, otherwise for each positive integer  $m \geq 1$  there is a radical ring  $R_m$  whose adjoint group is soluble of derived length  $n$  and there exist three sequences  $\{x_{mi} \mid i \geq 1\}$ ,  $\{y_{mi} \mid i \geq 1\}$ , and  $\{z_{mi} \mid i \geq 1\}$  of elements of  $R$  connected as in  $(*)$  such that  $x_{mm} \neq 0$ . Let  $R$  be the cartesian product of all these rings  $R_m$ . Then  $R$  is also a radical ring whose adjoint group is soluble of derived length  $n$ . On the other hand, if  $x_i, y_i$ , and  $z_i$  are elements of  $R$  whose  $m$ th components coincide with  $x_{mi}, y_{mi}$ , and  $z_{mi}$  for every  $i \geq 1$ , respectively, then the sequences  $\{x_i \mid i \geq 1\}$ ,  $\{y_i \mid i \geq 1\}$ , and  $\{z_i \mid i \geq 1\}$  satisfy condition  $(*)$  and  $x_i \neq 0$  for each  $i \geq 1$ , contrary to Lemma 3.1.

Thus, by Lemma 3.5, the ring  $R$  is Lie-soluble of length  $(m-1)n$  and there exists a finite chain

$$0 = I_0 \subseteq \cdots \subseteq I_i \subseteq I_{i+1} \subseteq \cdots \subseteq I_m = R$$

of ideals of  $R$  such that every factor  $I_{i+1}/I_i$  is generated by all commutative ideals of the factor ring  $R/I_i$  for  $0 \leq i \leq m-1$ , so that statements (1) and (2) are valid.

Conversely, let  $R$  be a radical ring for which the statements (1) and (2) hold. Then every factor  $I_{i+1}/I_i$  with  $0 \leq i \leq m-1$  is a radical ring whose adjoint group  $(I_{i+1}/I_i)^\circ$  is soluble of derived length at most  $l$  by Lemma 3.4. Since the adjoint groups  $I_i^\circ$  form a finite series of normal subgroups of  $R^\circ$  such that each factor  $I_{i+1}^\circ/I_i^\circ$  with  $0 \leq i \leq m-1$  is isomorphic to the adjoint group  $(I_{i+1}/I_i)^\circ$ , the group  $R^\circ$  is soluble of derived length at most  $(m-1)l$ . The theorem is proved.

*Proof of Theorem B.* Let  $x, y$  be two elements of  $R$ . As has been shown above, the Lie commutator  $[x, y]$  is nilpotent, so that there exists a least positive integer  $k = k(x, y, R, n)$  such that  $[x, y]^k = 0$ . Taking  $R$  as a cartesian product of a countable set  $\{R_i \mid i \geq 1\}$  of radical rings with two elements  $x_i, y_i$  of  $R_i$  and considering  $x, y$  as the elements whose  $i$ th component coincides with  $x_i$  and  $y_i$ , respectively, we find that  $k$  depends in fact only on  $n$ . By the same reason, for any elements  $x, y$  of the Levitzki radical  $L$  there exists a positive integer  $l = l(n)$  depending only on  $n$  such that  $[x, y, \dots, y] = 0$  whenever  $y$  appears  $l$  times. This means that  $L$  is an  $l$ -Engel ring and so the derived subgroup of  $R^\circ$  which is contained in  $L^\circ$  must be an  $m$ -Engel group for some  $m$  depending only on  $l$  by [3]. The theorem is proved.

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